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Uniqueness theorem for Inverse Sturm–Liouville Problem with Nonseparated Boundary Conditions[1]

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Consider the string, which vibrates in a medium with the variable elasticity coefficient $q(x)$. Interesting to follow the inverse problem: is it possible to determine the variable elasticity coefficient $q(x)$ by the natural frequencies of string vibrations. In 1946, G. Borg has been shown that a spectrum of frequencies is not sufficient to uniquely identify the medium elasticity coefficient $q(x)$. He offered the use of two frequency spectrum to uniquely identify of the medium elasticity coefficient $q(x)$. The second frequency spectrum is obtained by fastening the string to change at one of its ends to the other fastening. It was shown that these two frequency spectra already sufficient to uniquely identify $q(x)$ and the boundary conditions of both problems.

The case where the string fastening at one end depends on the other end fastening, is more difficult to solve. The boundary conditions, appropriate for the occasion, called nonseparated. Two spectra (of two boundary value problems) to restore both $q(x)$, and the nonseparated boundary conditions are not enough. In modern studies the spectra of the two eigenvalues boundary problems and an infinite sequence of signs is generally used for an uniqueness recovery. While this approach is useful in theoretical mathematics, it is inconvenient for the mechanics, because not clear the physical meaning of the corresponding sequence of signs.

In this article, instead of the two spectra and the sequence of signs as the spectral data are offered to use 7 of the eigenvalues of the initial boundary value problem, the spectrum, and the so-called norming constants of other boundary value problem. The physical sense of these data is quite clear. The first 7 eigenvalues of an initial boundary problem mean the first 7 natural frequencies of string vibrations. Norming constants represent norms from eigenfunctions. The spectrum and norming constants express a so-called spectral function. The spectral function gives a frequency spectrum with columns of vibrations amplitudes characteristics for string vibrations with other types of fastening.

Keywords: nonself-adjoint Sturm-Liouville problem, inverse problem, nonseparated boundary condition

1. Introduction

Nonself–adjoint Sturm–Liouville problem with nonseparated boundary conditions are considered. It is shown that this problem can be reconstructed by its seven eigenvalues and spectral data of an additional problem.

Let L denote the Sturm-Liouville problem

$$ly = -y'' + q(x)\,y = \lambda^2\,y, \tag{1}$$

$$U_1(y) = (a_{111} + a_{112}\,\lambda^2)\,y(0) + a_{121}\,y'(0) + \\ +(a_{131} + a_{132}\,\lambda^2)\,y(\pi) + a_{141}\,y'(\pi) = 0, \tag{2}$$

$$U_2(y) = (a_{211} + a_{212}\,\lambda^2)\,y(0) + a_{221}\,y'(0) + \\ +(a_{231} + a_{232}\,\lambda^2)\,y(\pi) + a_{241}\,y'(\pi) = 0, \tag{3}$$

where $q(x)$ is a continuous real valued function on the interval $[0,\pi]$; the coefficients a_{ijk} with $i,k = 1,2$, and $j = 1,2,3,4$ are real constants.

The inverse Sturm–Liouville problem for L in the case of separated boundary conditions have been well studied (see [1–3]). The inverse Sturm–Liouville problem with unknown coefficients in nonseparated boundary conditions was studied by V.A. Sadovnichii, V.A. Yurko, V.A. Marchenko, O.A. Plaksina,

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M.G. Gasymov, I.M. Guseinov, I.M. Nabiev, and other authors (see [4–7]).

The analysis of the inverse nonself–adjoint problem Sturm–Liouville with nonseparated boundary conditions was initiated in [5]. It was shown there that three spectra and two sets of weight numbers and residues of certain functions are sufficient for the unique reconstruction of a nonself–adjoint Sturm–Liouville problem with nonseparated boundary conditions. Moreover, these spectral data were used essentially [8].

Later, there were attempts to choose the problem to be reconstructed or auxiliary problems so as to use less spectral data for the reconstruction [4–7, 9]. In particular, in [9] a nonself-adjoint problem was replaced by a self-adjoint one, and it was shown that, for its unique reconstruction, as spectral data it suffices to use three spectra, some sequence of signs, and some real number. In [4], an auxiliary problem was chosen so as to reduce the number of spectral data required for the reconstruction of a self-adjoint problem by only two spectra, some sequence of signs, and some real number were used as spectral data. In [6] a nonself–adjoint Sturm–Liouville problem was uniquely reconstructed by two spectra and finite set of eigenvalues.

In the present paper, we consider more general nonself–adjoint Sturm–Liouville problem with nonseparated boundary conditions than in [6]. We show that, for its unique reconstruction, one can use also less spectral data as compared with the reconstruction of a self-adjoint problem in [4,5,9]; more precisely, we need finite set of eigenvalues of the nonself–adjoint Sturm–Liouville problem, and in addition, a spectrum and norming constants of an additional problem with separated boundary conditions.

Together with problem L, we consider the following problem with separated boundary conditions.

Problem L_1**:** The equation (1) with boundary conditions

$$U(y) := \lambda^2\,(y'(0) + h\,y(0)) + h_1\,y'(0) + h_2\,y(0) = 0,$$

$$V(y) := \lambda^2\,(y'(\pi) + H\,y(\pi)) + H_1\,y'(\pi) + H_2\,y(\pi) = 0,$$

where $-H_2 = a_{111}\,a_{221} - a_{121}\,a_{211}$, $h_1 = a_{112}\,a_{221} - a_{121}\,a_{212}$, $h_2 = a_{131}\,a_{221} - a_{121}\,a_{231}$, $H = a_{132}\,a_{221} - a_{121}\,a_{232}$, $H_1 = a_{111}\,a_{232} - a_{132}\,a_{211}$, $h = a_{112}\,a_{231} - a_{131}\,a_{212}$. The inverse problems for L_1 are studied in [10,11]. In [10] it is shown that the problem L_1 can be uniquely reconstructed by the spectral data of the problem L_1, and in [11] it is proved that the problem L_1 can be uniquely reconstructed by the Weyl function $M(\lambda)$ of the problem L_1. For the inverse problem of reconstructing problem L in which all coefficients a_{ijk} with $i, k = 1, 2$, and $j = 1, 2, 3, 4$ are unknown, no uniqueness theorems have been obtained.

In the present paper, we consider a nonself–adjoint Sturm–Liouville problem with nonseparated boundary conditions. We show that, for unique reconstruction of L we need seven eigenvalues of L and spectral data of an additional problem L_1.

We denote by $y_1(x,\lambda)$ and $y_2(x,\lambda)$ linearly independent solutions to differential equation (1) satisfying the conditions

$$\begin{aligned} y_1(0,\lambda) = 1, \quad y_1'(0,\lambda) = 0,\\ y_2(0,\lambda) = 0, \quad y_2'(0,\lambda) = 1. \end{aligned} \tag{4}$$

Eigenvalues λ_k of the problem L are roots of the characteristic determinant ([2, pp. 33–36], [3, pp. 29])

$$\Delta(\lambda) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix}. \tag{5}$$

The characteristic determinant $\Delta(\lambda)$ is an entire function. So, $\Delta(\lambda) \equiv 0$ and any value $\lambda \in \mathbb{C}$ is an eigenvalue; or $\Delta(\lambda) \not\equiv 0$ and the set of eigenvalues no more than countable [3, pp. 27].

We denote the matrix composed of the coefficients a_{lk} in the boundary conditions (2), (3) by A:

$$A = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{matrix} \right\| \tag{6}$$

and its minors composed of the ith and jth columns by M_{ij}:

$$M_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix},$$

where $a_{i1} = a_{i1}(s) = a_{i11} + a_{i12}\,\lambda^2$, $a_{i2} = a_{i21}$, $a_{i3} = a_{i3}(s) = a_{i31} + a_{i32}\,\lambda^2$, $a_{i4} = a_{i41}$, $i = 1, 2$.

Let the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of equation (1) under the initial conditions

$$\varphi(0,\lambda) = h_1 - \lambda^2, \qquad \varphi'(0,\lambda) = h_2 - \lambda^2\,h, \tag{7}$$

$$\psi(\pi,\lambda) = H_1 - \lambda^2, \quad \psi'(\pi,\lambda) = \lambda^2\,H - H_2. \tag{8}$$

By definition, put $\chi(\lambda) = \varphi(x,\lambda)\,\psi'(x,\lambda) - \varphi'(x,\lambda)\,\psi(x,\lambda)$, which is independent of $x \in [0,\pi]$. The function $\chi(\lambda)$ is entire and has zeros at the eigenvalues μ_n of the problem L_1. The set of eigenvalues μ_n of the problem L_1 is countable, consists of real numbers and for each eigenvalue μ_n there exists such a number $k_n \neq 0$ that

$$\psi(x,\mu_n) = k_n\,\varphi(x,\mu_n).$$

Numbers γ_n are *norming constants* if [10]:

$$\gamma_n = \int_0^\pi \varphi^2(x,\mu_n)\,dx + \delta + \frac{\rho}{k_n^2}.$$

The numbers $\{\mu_n,\,\gamma_n\}$ are called *spectral data* of problem L_1.

For problem L, we pose the inverse problem.

2. Inverse Problem

Suppose that the potential function $q(x)$ and the coefficients in the boundary conditions of problem L are unknown, while the spectral data $\{\mu_n, \gamma_n\}$ of the problem L_1 and seven eigenvalues λ_m ($m = 1, 2, 3, 4, 5, 6, 7$) of the problem L are known. It is required to find $q(x)$ and boundary conditions of the problem L from the spectral data $\{\mu_n, \gamma_n\}$ of the problem L_1 and seven eigenvalues λ_m ($m = 1, 2, 3, 4, 5, 6, 7$) of the problem L.

In [10] author proved that if $h\,h_1 - h_2 > 0$, $H\,H_1 - H_2 > 0$, then the boundary value problem L_1 is uniquely determined by the spectral data $\{\mu_n, \gamma_n\}$ of the problem L_1.

Let λ_m ($m = 1, 2, 3, 4, 5, 6, 7$) be seven eigenvalues of problem L.

Theorem. *If the spectrum of the problem L is infinity, then problems L and L_1 can be uniquely reconstructed from the spectral data $\{\mu_n, \gamma_n\}$ of the problem L_1 and seven eigenvalues λ_m ($m = 1, 2, 3, 4, 5, 6, 7$) of the problem L such that*

$$D = \det\Big(\big\|1, \lambda_m^2, y_2'(\pi, \lambda_m), \lambda_m^2\, y_2'(\pi, \lambda_m), y_1'(\pi, \lambda_m), y_2(\pi, \lambda_m), \lambda_m^4\, y_2(\pi, \lambda_m)\big\|_{m=1,2,\ldots,7}\Big) \neq 0.$$

Proof. It follows from [10] that if $h\,h_1 - h_2 > 0$, $H\,H_1 - H_2 > 0$, then the function $q(x)$ and the numbers h, h_1, h_2, H, H_1, and H_2 is uniquely determined by the spectral data $\{\mu_n, \gamma_n\}$ of the problem L_1. Since spectrum of the problem L is infinity, we see that the seven eigenvalues λ_m ($m = 1, 2, 3, 4, 5, 6, 7$) of the problem L exist. Let us show that x_m ($m = 1, 2, \ldots, 7$) uniquely determined by seven eigenvalues λ_m ($m = 1, 2, \ldots, 7$). Suppose

$$D = \det\Big(\big\|1, \lambda_m^2, y_2'(\pi,\, \lambda_m), \lambda_m^2\, y_2'(\pi,\, \lambda_m), y_1'(\pi,\, \lambda_m), y_2(\pi,\, \lambda_m), \lambda_m^4\, y_2(\pi,\, \lambda_m)\big\|_{i=1,2,\ldots,7}\Big) \neq 0.$$

Then the numbers λ_m ($m = 1, 2, \ldots, 7$) are eigenvalues of the problem L, so $\Delta(\lambda_m) = 0$. It now follows that

$$\begin{gathered} x_1 + x_2\lambda_m^2 + (x_3 + x_4\lambda_m^2) y_2'(\pi, \lambda_m) + \\ + x_5 y_1'(\pi, \lambda_m) + (x_6\lambda_m^2 + x_7\lambda_m^4) y_2(\pi, \lambda_m) \\ = H_2 + h_1\,\lambda_m^2 - (h_2 + H\,\lambda_m^2)\, y_1(\pi,\, \lambda_m) - \\ -(H_1 + h)\,\lambda_m^2. \end{gathered} \tag{9}$$

The determinant D of system (9) with respect to the unknowns x_m ($m = 1, 2, \ldots, 7$) is the 7th–order Vandermonde determinant equal to $\prod_{m_1 > m_2}(\lambda_{m_1} - \lambda_{m_2}) \neq 0$. Therefore, the system (9) has a unique solution, which can be found by the Cramer formulas. It follows from this (see [12]) that the matrix A is determined up to a linear transformation of the rows by the minors h, h_1, h_2, H, H_1, H_2, and x_m ($m = 1, 2, \ldots, 7$). Hence the problems L and L_1 are uniquely determined by the spectral data $\{\mu_n, \gamma_n\}$ of the problem L_1 and seven eigenvalues λ_m ($m = 1, 2, 3, 4, 5, 6, 7$) of the problem L. This completes the proof of the theorem.

3. Conclusions

In this paper it is shown that the medium elasticity coefficient $q(x)$ and the coefficients in the boundary conditions of problem L can be uniquelly reconstructed by seven eigenvalues of Problem L and the spectral data $\{\mu_n, \gamma_n\}$ of the additional problem L_1. Erlier for the problem of reconstructing problem L in which all coefficients a_{ijk} with $i, k = 1, 2$, and $j = 1, 2, 3, 4$ are unknown, no uniqueness theorems have been obtained. Only solutions for special cases of this inverse problem are known. However, even in this special cases of problem L for unique reconstruction of the medium elasticity coefficient $q(x)$ and the coefficients in the boundary conditions of problem L, we can use less spectral data as compared with the papers of other authors. We need finite set of eigenvalues of L instead of some infinite sequence of signs and some real number.

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